

# TECHNICAL RESEARCH REPORT

## Stabilization of LTI Systems with Communication Constraints

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# Stabilization of LTI Systems with Communication Constraints<sup>1</sup>

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## Abstract

This work is directed towards exploring interactions of communication and control in systems with communication constraints. Examples of such systems include groups of autonomous vehicles, MEMS arrays and systems whose sensors and actuators are distributed across a network. We extend some recent results involving stabilization of LTI systems under limited communication and address a class of feed-forward control problems for the systems of interest.

## 1 Introduction

With communication networks proliferating into increasingly many aspects of modern-day technology, engineers are faced with new opportunities involving the control of systems whose components are distributed across a network and which are supposed to operate in a coordinated manner. Examples of such systems include groups of vehicles, satellite clusters, mobile communications, MEMS arrays and others. The challenges that these systems present go beyond problems associated with increased dimensionality. Frequently, their performance is limited not because of lack of computational power but because of lack of time on a shared network of sensors and actuators. This has led to recent efforts towards bringing together aspects of control and communication under a framework that will lead to a better understanding of control systems with communication constraints.

One way to jointly formulate control and communication problems is to employ the idea of a “communication sequence” [6] which allows multiple (sub)systems to share the attention of a centralized controller [8]. Communication sequences bring forth some of the connections between control, communication and compu-

tational complexity and are appealing because they quantify the amount of “attention” that the controller pays to each component of the underlying system. Previous work has addressed problems involving stabilizing an LTI system when only some of its output(s) can be measured at one time [10]. That work explored the effects of communicating sequentially with each of a set of linear systems and described a method for stabilizing all systems in the set. This paper extends and completes those results, taking into account communication constraints that are present both when making measurements as well as when transmitting control actions to the LTI system.

We present a new “extensification algorithm” that transforms the stabilization problem to an equivalent problem involving matrix search (as is the case with the work in [10]). Our goal is to combine our algorithm together with recent results on tracking for networked control systems ([8]) so that closed-loop tracking can be achieved for the systems of interest. For related problems in hybrid system theory see [3], [5] [4], and [12]. Work discussing systems with communication constraints can be found in [2], [11] and others.

## 2 A Prototype Computer-Controlled System

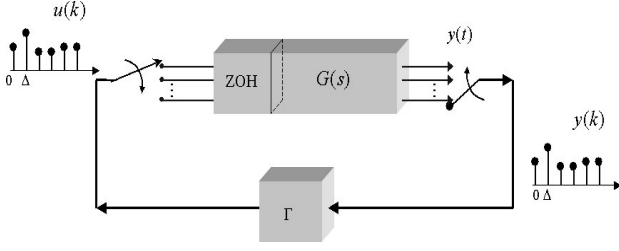
Consider an  $n$ -dimensional LTI system  $G(s)$  with input  $u \in \mathbb{R}^m$ , and output  $y \in \mathbb{R}^p$ . The system is driven by a computer or other digital controller (Fig. 1) that may be remotely located and which does not have simultaneous access to all inputs/outputs of the control system. In particular:

- The controller sends commands to and receives inputs from the system every  $\Delta$  units of time, via a zero-order-hold stage.
- Inputs/outputs are communicated through a bus which has limited capacity. Specifically, the bus can “carry” at most  $b > 0$  signals, with  $b < m + p$ ,  $b \in \mathbb{N}^*$ .

Typically, the capacity of the communication bus will be divided between input and output signals, with  $b_r$  channels devoted to sampling the output of

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**Figure 1:** A closed-loop computer-controlled system

the underlying LTI system and  $b_w$  channels used for transmitting control inputs. We will refer to these two groups of channels as the “input” and “output bus.” Of course,  $b_r$  and  $b_w$  may change at any time as long as  $b_r + b_w = b$ . This represents a rather general setting for controller/plant communication, allowing for dynamic reconfiguration of the available channels. We will use the terms “narrow” ( $b_w < m$  or  $b_r < p$ ) and “wide” ( $b_w \geq m$  and  $b_r \geq p$ ) to describe the communication bus.

In the following it will be convenient to use the discrete-time equivalent of the linear system  $G(s)$ :

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \quad (1)$$

Each element of  $u$  retains its value (by virtue of the ZOH stage) until that element is updated by the controller. At the same time, the controller may only receive partial information about the output  $y(k)$ . When the communication bus is narrow, one possibility is to choose a sequence of operations for the switches (see Fig. 1) that select which inputs/outputs are to be updated/sampled at a particular time. This leads us to the notion of a “communication sequence” (originally introduced in [6]) by which the controller chooses which of the input signals (measurements) to update (read) at every step.

**Definition 1** An  $N$ -periodic communication sequence is an element of

$$\mathbb{E}_{per}^{m \times N} = \{(\sigma(0), \sigma(1), \dots, \sigma(N-1), \dots, \sigma(N-1), \dots) : \sigma(i) \in \{0, 1\}^m\} \quad (2)$$

for some  $m > 0$ .

We will consider the controller’s communication with the system to follow a periodic pattern specified by a pair of  $N$ -periodic sequences: a “control” sequence  $\sigma_w \in \mathbb{E}_{per}^{m \times N}$  will be used to transmit inputs and a “measurement” sequence  $\sigma_r \in \mathbb{E}_{per}^{p \times N}$  will provide a pattern for sampling the system output. The entries of  $\sigma_w(i)$  ( $\sigma_r(i)$ ) indicate which elements of  $u(k)$  ( $y(k)$ ) are to be updated (measured) at the  $k^{th}$  time step. We will ignore quantization errors associated with the representation of signal samples in the digital controller

and with the transmission of those samples through the communication bus.

**Definition 2** Consider a computer-controlled system  $G(z)$  with  $b_w < b$  ( $b_r < b$ ) being the dimension of the input (output) communication bus. A pair of communication sequences  $\sigma_w \in \mathbb{E}^{m \times N}$ ,  $\sigma_r \in \mathbb{E}^{p \times N}$  is **admissible** if:

- $\|\sigma_w(i)\|^2 \leq b_w, \|\sigma_r(i)\|^2 \leq b_r \quad \forall i = 0, \dots, N-1$
- $\sum_{j=1}^m \sigma_w(i) + \sum_{j=1}^p \sigma_r(i) \leq b \quad \forall i = 0, \dots, N-1$
- $\text{Span}\{\sigma_w(0), \dots, \sigma_w(N-1)\} = \mathbb{R}^m$  and  $\text{Span}\{\sigma_r(0), \dots, \sigma_r(N-1)\} = \mathbb{R}^p$

The above conditions require that no more than  $b_w$  ( $b_r$ ) of the system inputs (outputs) are updated (measured) by the controller at every step and that the pair  $(\sigma_w, \sigma_r)$  allows communication with all inputs (outputs) of the linear system at least once every period.

### 3 Stabilization with Limited Communication

We now focus on the problem of stabilizing systems like the one discussed in the previous section, using static output feedback (see Fig. 1). We will take the number of input and output channels ( $b_w$  and  $b_r$ ) to be constant.

**Problem Statement 1** Given: a computer-controlled system  $G(z)$  with  $b_r, b_w \in N^*$ ,  $b_r + b_w = b$  denoting the size of the input and output communication busses and a pair of admissible  $N$ -periodic communication sequences  $\sigma_r, \sigma_w \in \mathbb{E}_{per}^{p \times N}, \mathbb{E}_{per}^{m \times N}$ , find a constant feedback gain  $\Gamma \in \mathbb{R}^{p \times m}$  that stabilizes the system under output feedback.

In [10], we showed that a version of this problem (with no constraints on transmitting control samples  $u(k)$ ) is equivalent to the NP-hard problem:

**Problem Statement 2** Given a collection of matrices  $A_i \in \mathbb{R}^{q \times q}$ ,  $0 \leq i \leq i_{max}$ , and scalars  $\gamma_1, \dots, \gamma_{i_{max}} \in \mathbb{R}$ , find a stable element of the affine subspace

$$\mathcal{A} = \mathcal{A}_0 + \sum_{i=1}^{i_{max}} \gamma_i \mathcal{A}_i \quad (3)$$

where  $q = (2N^2 - N)n$  and  $i_{max} = mp$ . In the next section we present a new extensification algorithm. This algorithm is complete in the sense that takes into account the both input ( $b_w < m$ ) and measurement constraints ( $b_r < p$ ) and arrives at a similar construction for the matrices that span the affine subspace of interest.

### 4 Extensive Form of a Discrete LTI System

Consider the system of Fig. 1, to which a static output feedback controller is attached, subject to limited

communication as described in Sec. 2. If the input and output busses were both “wide” ( $b_r = p$ ,  $b_w = m$ ) then the static output feedback control  $u(k) = \Gamma C x(k)$  would be possible to implement and  $\Gamma$  would have to be chosen to make the closed-loop dynamics  $(A + B\Gamma C)$  stable. We proceed to modify this simple situation to reflect the existence of communication constraints.

#### 4.1 Constrained Measurements

Assume for now that  $b_w = m$  so that the controller can transmit the entire input vector at once to the LTI system. Because measurements are to be obtained according to the communication sequence  $\sigma_r \in \mathbb{E}_{per}^{p \times N}$ , only some of the elements of  $y(k)$  are received by the controller at each step  $k$ . The elements for which the controller does not receive updates are assumed to hold their last-known values. More precisely, we have a control law of the form

$$u(k) = \Gamma y_l(k) \quad (4)$$

where  $y_l(k)$  is the output vector composed of the most up to date information available to the controller at the  $k^{th}$  step. Notice that in general  $y_l(k) \neq y(k)$  because not all elements of  $y(k)$  are communicated to the controller at step  $k$ . We can write

$$y_l(k) = \text{diag}(\sigma_r(k))y(k) + (I - \text{diag}(\sigma_r(k)))y_l(k-1) \quad (5)$$

where for a vector  $x \in \mathbb{R}^n$ ,  $\text{diag}(x)$  is an  $n \times n$  matrix with the elements of  $x$  along its diagonal and all off-diagonal entries being zero. By iteratively applying Eq. 5 for a number of steps equal to the communication period  $N$ , we obtain

$$y_l(k) = \text{diag}(\sigma_r(k))y(k) + \sum_{i=1}^{N-1} \text{diag}(\sigma_r(k-i)) \left( \prod_{j=0}^{i-1} M_R(k,j) \right) y(k-i) \quad (6)$$

where  $M_R(k,j) \triangleq I - \text{diag}(\sigma_r(k-j))$ . We observe that if the communication sequence  $\sigma_r$  is admissible then each of the  $p$  elements of the output  $y(k)$  will be read at least once every  $N$  steps so the summation in Eq. 6 terminates after at most  $N$  steps: system outputs older than  $N$  steps are essentially “overwritten” in the controller. We can rewrite Eq. 6 as

$$y_l(k) = \sum_{i=0}^{N-1} D_R(k,i) C x(k-i) \quad (7)$$

where

$$D_R(k,i) \triangleq \begin{cases} \text{diag}(\sigma_r(k)) & i = 0 \\ \text{diag}(\sigma_r(k-i)) \prod_{j=0}^{i-1} M_R(k,j) & i > 0 \end{cases} \quad (8)$$

are diagonal  $p \times p$  matrices with binary entries. The  $j^{th}$  diagonal element of  $D_R(k,i)$  is 1 if the  $j^{th}$  output was last read at the  $(k-i)^{th}$  step and is 0 otherwise.

#### 4.2 Constrained Control

We now follow the procedure of Sec. 4.1, applied this time to communication over the input bus. The input vectors  $u(k)$  arrive at the LTI system according to the communication sequence  $\sigma_w \in \mathbb{E}_{per}^{m \times N}$ . At the  $k^{th}$  step, only the inputs specified by the non-zero entries of  $\sigma_w(k)$  are updated, with all other inputs remaining at their previous values:

$$u(k) = \text{diag}(\sigma_w(k))\Gamma y_l(k) + (I - \text{diag}(\sigma_w(k)))u(k-1) \quad (9)$$

By iterating backwards for a full period ( $N$  steps) and assuming that the communication sequence  $\sigma_w$  is admissible (i.e. all  $m$  elements of the input are updated at least once every  $N$  steps) we obtain:

$$u(k) = \sum_{i=0}^{N-1} D_W(k,i)\Gamma y_l(k-i) \quad (10)$$

where

$$D_W(k,i) \triangleq \begin{cases} \text{diag}(\sigma_w(k)) & i = 0 \\ \text{diag}(\sigma_w(k-i)) \prod_{j=0}^{i-1} M_W(k,j) & i > 0 \end{cases} \quad (11)$$

are diagonal  $m \times m$  binary matrices and  $M_W(k,j) \triangleq I - \text{diag}(\sigma_w(k-j))$ . The  $j^{th}$  diagonal element of  $D_W(k,i)$  is 1 if the  $j^{th}$  input was last updated at the  $(k-i)^{th}$  step, 0 otherwise.

#### 4.3 Combining Communication Constraints

If communication with the system is to proceed according to the pair  $(\sigma_r, \sigma_w)$  of admissible  $N$ -periodic sequences, we can combine the results of Sec. 4.1 and Sec. 4.2. By substituting Eq. 7 into Eq. 10 we obtain:

$$Bu(k) = \sum_{i=0}^{2N-2} F_{ki} x(k-i) \quad (12)$$

where

$$F_{ki} \triangleq B \sum_{j=\min(i,N-1)}^{\lfloor \frac{i}{N} \rfloor (i-N+1)} D_W(k,j)\Gamma D_R(k-j,i-j)C \quad (13)$$

We can now write the closed-loop dynamics of the computer-controlled system as:

$$x(k+1) = Ax(k) + \sum_{i=0}^{2N-2} F_{ki} x(k-i) \quad (14)$$

Let us define  $\text{Comp}(p)$  to be the companion form associated with an  $n^{th}$ -degree polynomial  $p(s) = \sum_0^n p_i s^i$

$$\text{Comp}(p) \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ p_n & p_{n-1} & \cdots & p_1 & p_0 \end{bmatrix} \quad (15)$$

If we now use the  $F_{kj}$  (Eq. 13) to define the matrix polynomial

$$f_k(s) = A + \sum_{i=0}^{2N-2} F_{kj} s^i \quad (16)$$

then the closed-loop dynamics of Eq. 14 can be expressed in first-order form:

$$\chi(k+1) = \text{Comp}(f_k)\chi(k) \quad (17)$$

where  $\chi = [x_{(k-2N+1)}^T \cdots x_{(k)}^T x_{(k+1)}^T]^T \in \mathbb{R}^{(2N-1)n}$ .

The system of Eq. 17 is linear, time-varying and describes the state evolution of the computer-controlled system under output feedback and periodic communication. We have essentially “extensified” the state vector to include past values up to two communication periods. We note that the matrix  $\text{Comp}(f_k)$  is  $N$ -periodic in  $k$ , so Eq. 17 represents a periodic linear system of dimension  $(2N-1)n$ . Although of larger dimension, this periodic system is equivalent to the original system in the sense that it exactly describes the closed loop dynamics under the communication policy that was imposed.

It is a fact that every discrete-time periodic system can be expressed as a time-invariant system of higher dimension (see [7]). In our case, this yields a system of order  $(2N^2 - N)n$  which we call the “extensive form” of the original system in Problem 1:

$$\mathcal{X}_e(k+1) = \mathcal{A}\mathcal{X}_e(k) \quad (18)$$

where  $\mathcal{X}_e(k) \in \mathbb{R}^{(2N^2-N)n}$  and

$$\mathcal{A} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \text{Comp}(f_0) \\ \text{Comp}(f_1) & 0 & \cdots & 0 & 0 \\ 0 & \text{Comp}(f_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \text{Comp}(f_{N-1}) & 0 \end{bmatrix} \quad (19)$$

By construction, stability of the extensified system (Eq. 18) is equivalent to the stability of the original system. Moreover, each of the matrices  $\text{Comp}(f_k)$  are affine in the entries of  $\Gamma$ . By choosing a basis for  $\mathbb{R}^{m \times p}$ , we can express  $\Gamma$  as  $\Gamma = \sum_{i=0}^{mp} \gamma_i E_i$  where  $E_i$  is an  $m \times p$  matrix whose  $(\lfloor \frac{i}{p} \rfloor + 1, i \bmod p + 1)^{th}$  entry is “1”, with all other entries being zero. In the basis of the  $\{E_i\}$  we can express  $\mathcal{A}$  as an element of the affine subspace

$$\mathcal{A} = \mathcal{A}_0 + \sum_{i=0}^{mp} \gamma_i \mathcal{A}_i \quad (20)$$

where each of the  $\mathcal{A}_i$  are obtained by substituting  $E_i$  for  $\Gamma$  in Eq. 13.

We note that at the  $k^{th}$  step, the latest available output to the controller  $y_l(k)$  depends on the past  $N$  values

of the state. If all elements of  $u(k)$  could be communicated simultaneously (wide input bus), it would be sufficient to “extensify” the state by considering only its  $N$ -past values at times  $k, k-1, \dots, k-N-1$  (see [10]). However, because the inputs are also subject to delays in arriving at the LTI system, a particular element of  $u(k)$  could persist for (at most) the next  $N$  steps  $k, k+1, \dots, k+N-1$ . This implies that the entries of  $u(k)$  as seen by the LTI system could depend on state values that are (at most)  $2N-1$  steps old.

In summary, we have given a procedure for converting an output feedback stabilization problem involving LTI systems under limited communication, into a search problem involving a finite collection of  $(2N^2 - N)n$ -dimensional matrices. These matrices are obtained from the parameters of the control system together with a pair of admissible communication sequences.

Equations 18 and 19 show that after a full communication period, the stability properties of the extensive form can be captured in a lower-dimensional space by considering

$$z(k+1) = \hat{\mathcal{A}}z(k) \quad (21)$$

where  $\hat{\mathcal{A}} = \mathcal{A}_{N-1}\mathcal{A}_{N-2}\dots\mathcal{A}_0$  and  $\dim(z) = (2N-1)n$ . In this lower-dimensional space, the number of gains is the same as in the extensive form, however the gains enter  $\hat{\mathcal{A}}$  nonlinearly. If the number of gains is  $n_\gamma \triangleq mp$  then the number of matrix coefficients for the extensive form is  $n_\gamma + 1$ . In the lower-dimensional space the number of matrix coefficients would be  $\sum_{i=0}^N \binom{n_\gamma+i-1}{i}$ . Thus, even though lower-dimensional matrices are often convenient from a computational viewpoint, the potential savings in algorithm run times are offset by the memory demands due to the large number of coefficient matrices that Eq. 21 requires.

## 5 An Extensification Example

Consider the scalar system

$$x(k+1) = ax(k) + u(k) \quad (22)$$

$$y(k) = x(k) \quad (23)$$

We assume that the controller communicates with the above system over a bus of width  $b = 1$  according to the following pair of 2-periodic communication sequences:

$$\sigma_r = (1, 0, 1, 0, \dots), \quad \sigma_w = (0, 1, 0, 1, \dots) \quad (24)$$

Notice that the only communication channel is used for measurements of the output and for transmission of the control actions in an alternating fashion. Clearly, the above sequences are admissible. We want to stabilize the system using a control law of the form

$$u(k) = \gamma y(k - d(\sigma_r, \sigma_w, k)) \quad (25)$$

where  $d(\sigma_r, \sigma_w, k)$  is a delay that depends on the communication sequences and the current step  $k$ .

The scalar system we began with, combined with the period-2 pair of communication sequences  $\sigma_r, \sigma_w$  gives rise to a 3-dimensional periodic system  $\chi(k+1) = \text{Comp}(f_k)\chi(k)$  with

$$f_k(s) = \begin{cases} a + \gamma s^2 & k \text{ even} \\ a + \gamma s & k \text{ odd} \end{cases} \quad (26)$$

The corresponding companion forms are:

$$\text{Comp}(f_k) = \begin{cases} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & 0 & a \end{bmatrix} & k \text{ even} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \gamma & a \end{bmatrix} & k \text{ odd} \end{cases} \quad (27)$$

and the extensive form is given by the 6-dimensional LTI system:

$$\mathcal{X}_e(k+1) = \left[ \begin{array}{ccc|ccc} & & & 0 & 1 & 0 \\ & 0 & & 0 & 0 & 1 \\ & & & \gamma & 0 & a \\ \hline 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & 0 \\ 0 & \gamma & a & & & \end{array} \right] \mathcal{X}_e(k) \quad (28)$$

Choosing the gain  $\gamma$  in order to stabilize the above system is equivalent to finding a stable element of the form given by Eq. 20 where  $\mathcal{A}_0$  and  $\mathcal{A}_1$  can be read off from Eq. 28.

## 6 Finding a Set of Stabilizing Gains

After a computer-controlled system has been put in the extensive form as described in Sec. 4, the output stabilization problem becomes equivalent to finding a stable element of the affine subspace defined by Eq. 20. This is an NP-hard problem [1]. One possibility is to choose the gains  $\gamma_i$  so that the eigenvalues of  $\mathcal{A} = \mathcal{A}_0 + \sum_i \gamma_i \mathcal{A}_i$  are enclosed in a circle with the smallest possible radius. This suggests minimizing the spectral radius of the closed-loop system

$$\eta = \|\lambda_{\max}(\mathcal{A})\| \quad (29)$$

To negotiate the large number of local minima that are expected, we use simulated annealing on the gains  $\gamma_i$ . Our algorithm numerically computes the gradient  $\partial\eta/\partial\gamma_i$  and then lets the gains  $\gamma_i$  flow along that gradient, adding a white-noise term  $dw$  with a gain  $g(t)$  that decays to zero:

$$d\gamma_i = \frac{\partial\eta}{\partial\gamma_i} dt + g(t)dw. \quad (30)$$

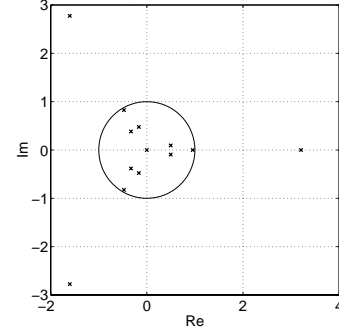
The “cooling schedule”  $g(t)$  should go to zero as  $t \rightarrow \infty$ , but it should do so at a slow enough rate for the spectral radius to approach the global minimum.

## 7 Simulation Results

Consider the fourth-order LTI system:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 3/4 & 1/2 & 0 \\ 1/4 & 3 & 1/3 & -1/3 \\ 1/6 & 0 & -1/2 & -3/7 \\ 0 & -1 & 2/5 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} x \end{aligned} \quad (31)$$

whose open-loop system eigenvalues are shown in Fig. 2. We want to stabilize this LTI system using



**Figure 2:** Eigenvalues of open-loop system,  $\|\lambda_{\max}\| = 3.2$ .

static output feedback, given that the communication bus can carry two signals to/from any of the inputs or outputs (i.e.  $b = 2$ ). Of the two available channels, one is to be used for transmitting control values, the other for obtaining measurements. In the following, we investigate the performance of the extensification and simulated annealing algorithms (Sec. 4, 6) for two different communication sequences.

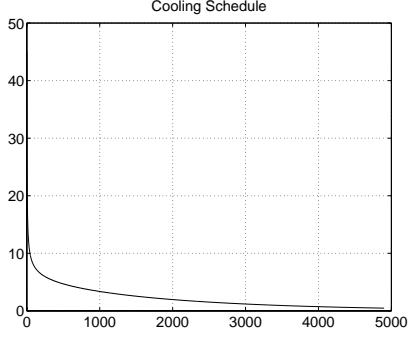
### 7.1 Control with Uniform Attention

Initially the controller is to divide its attention equally between each of the input/output pairs  $(u_1, y_1)$  and  $(u_2, y_2)$ . For that purpose, we chose

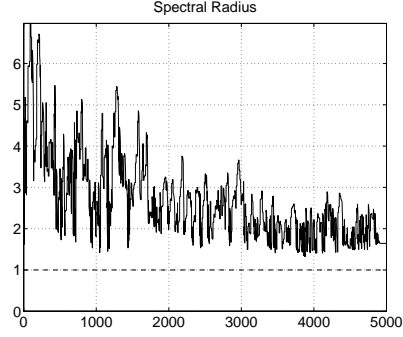
$$\sigma_w = \sigma_r = \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \dots \right) \quad (32)$$

to be the communication sequences to be used, so that the controller reads and transmits in an alternating pattern.

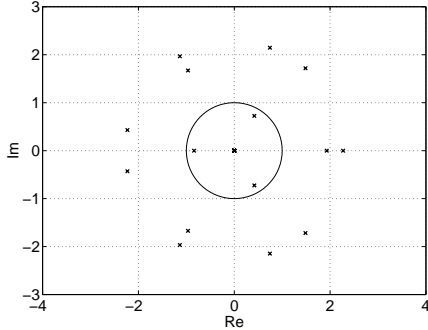
The matrices composing the extensive form were computed and simulated annealing was performed on the four elements of the feedback matrix  $\Gamma$ . The simulated annealing algorithm was stopped after 5000 steps. The cooling schedule was a  $\frac{1}{\log(n)}$  decay followed by a linear decay to zero when the gain  $g(t)$  reached a pre-specified level of 0.1. A plot of the cooling schedule is shown in Fig. 3. Choosing “good” cooling schedules for the stabilization problem considered here remains an open problem. In this case, the spectral radius did not reach values below unity. The evolution of the spectral radius is shown in Fig. 4. The resulting final eigenvalues of the extensified system are shown in Fig. 5, corresponding to a closed-loop system that was unstable.



**Figure 3:** Cooling schedule.



**Figure 4:** Evolution of spectral radius (uniform attention).



**Figure 5:** Closed-loop eigenvalues with uniform attention,  $\|\lambda_{max}\| = 2.7$ .

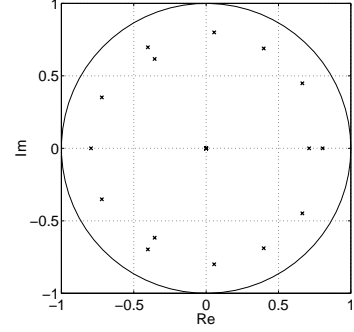
## 7.2 Towards Optimal Communication

Next, we investigated a period-four pair of communication sequences,

$$\sigma_w = \sigma_r = \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \dots \right) \quad (33)$$

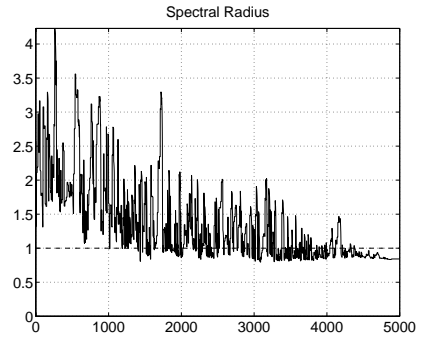
that devote three cycles to the pair  $(u_1, y_1)$  for every one cycle allocated to  $(u_2, y_2)$ . The above sequences were chosen after some experimentation and by noticing that the upper-left  $2 \times 2$  block of the dynamics for the state evolution (Eq. 31) has a larger spectral radius than the lower-right block (when the coupling between the two blocks is removed). As a result, communicating more often with the  $(u_1, y_1)$  pair may lead to better performance.

The cooling schedule was the same as in the uniform attention case. This time, simulated annealing stabilized the closed-loop system, reducing the spectral radius to 0.795. The final closed-loop eigenvalues are shown in Fig. 6. with the evolution of the spectral radius of the



**Figure 6:** Closed-loop eigenvalues with non-uniform attention,  $\lambda_{max} = 0.795$ .

system shown in Fig. 7.



**Figure 7:** Evolution of spectral radius (non-uniform attention).



## 8 Combining Stabilization with Feed-forward Control

We outline a method for combining stabilization and feed-forward control. Consider an LTI system subject to communication constraints as described in Sec. 2 and assume that a stabilizing feedback matrix  $\Gamma$  has been computed. Let  $\{y_d\}_{k=1}^{k_m}$ ,  $(y_d(k) \in \mathbb{R}^p, k_m > 0)$  be a desired output sequence of finite duration, which we want to track with the output of the system of Fig. 1. That is, we allow the controller to send signals of the form

$$u(k) = \Gamma y(k - d(\sigma_w, \sigma_r, k)) + u'(k) \quad (34)$$

where  $d()$  is a delay due to the communication constraints. The signal  $u'(k)$  is a feed-forward term which is subject to the same communication constraints  $(\sigma_r, \sigma(w))$  as the feedback signal so that it may be superimposed on the stabilizing inputs transmitted by the controller as Eq. 34 shows. We want to select the sequence  $u'(k)$ ,  $k = 0, \dots, k_m$  so that the error:

$$\sum_{i=0}^{k_m} \|y_d(k) - y(k)\| \quad (35)$$

is minimized. In the following, we assume that the duration  $(k_m)$  of the desired signal is a multiple of  $(2N - 1)$ .

If the desired output  $y_d(k)$  is known a-priori, then we can make use of operator-based approaches [8] for this “preview tracking” problem, suitably modified for discrete-time systems. In short, consider the LTI system discussed in Sec. 2, in its periodic form of Eq. 17

$$\chi(k+1) = A_c(k)\chi(k) \quad (36)$$

with  $A_c(k) = \text{Comp}(f_k)$   $N$ -periodic, defined by Eq. 15, 16. The dynamics of the stabilized system (Eq. 36) must be modified to reflect the presence of the feed-forward signal  $u'(k)$ . Let

$$\begin{aligned} v(k) &\triangleq [u_{(k-2N+2)}^T \cdots u_{(k-1)}^T u_{(k)}^T]^T \\ \psi(k) &\triangleq [y_{(k-2N+2)}^T \cdots y_{(k-1)}^T y_{(k)}^T]^T \end{aligned} \quad (37)$$

then

$$\begin{aligned} \chi(k+1) &= A_c(k)\chi(k) + B_c(k)v(k) \\ \psi(k) &= C_c(k)\chi(k) \end{aligned} \quad (38)$$

where  $B_c \in \mathbb{R}^{(2N-1)n \times m}$  and  $C_c \in \mathbb{R}^{(2N-1)p \times (2N-1)n}$  will be determined by the choice of communication sequence and the parameters of the underlying liner system.

Now, feed-forward input sequences are mapped to outputs by

$$\tilde{\psi} = \Lambda \tilde{v} \quad (39)$$

where

$$\tilde{\psi} \triangleq \begin{bmatrix} \psi(1) \\ \psi(2) \\ \vdots \\ \psi(\mu) \end{bmatrix} \quad \tilde{v} = \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(\mu-1) \end{bmatrix}, \quad (40)$$

$\mu = k_m/(2N - 1)$  and

$$\Lambda \triangleq [\Lambda_{ij}] \quad (41)$$

with each  $(2N - 1)p \times (2N - 1)m$  block  $\Lambda_{ij}$  given by

$$\Lambda_{ij} = \begin{cases} C_c(i) \left( \prod_{q=i}^j A_c(q) \right) B_c(j) & i > j \\ C_c(i) B_c(j) & i = j \\ 0 & j > i \end{cases} \quad (42)$$

If  $p \geq m$ ,  $\Lambda$  has more rows than columns. If  $\Lambda$  has maximal column-rank, then we can use its generalized inverse to compute the feed-forward control sequence that will minimize the tracking error (Eq. 35) by:

$$v^* = (\Lambda^T \Lambda)^{-1} \Lambda^T (\psi_d - \psi_{ic}) \quad (43)$$

where  $\psi_{ic}$  is the effect of initial conditions for the plant state  $x(0)$ . The vector  $\psi_{ic}$  will be comprised of the samples of the sequence

$$y_{ic}(k) = CA^k x(0) \quad (44)$$

according to Eq. 37, 40.

There are two necessary conditions for  $\Lambda$  to be injective. First, the original system (Eq. 1) have normal rank:

$$\limsup_{|z|=1} \text{rank}(C(zI - A)^{-1}B) = m \quad (45)$$

Second, the communication sequences  $\sigma_r, \sigma_w$  must be admissible, to prevent from consistently “ignoring” any of the inputs or outputs. Although these are only necessary conditions, it is possible to modify matters so that they are also sufficient, by redefining  $\tilde{v}(k)$  in Eq. 39 to be:

$$\tilde{v} = \begin{bmatrix} v(-2N-1) \\ \vdots \\ v(-1) \\ v(0) \\ v(1) \\ \vdots \\ v(\mu-1) \end{bmatrix} \quad (46)$$

and  $\Lambda$  to be the new map that takes  $\tilde{v}$  to  $\tilde{y}$ . This corresponds to considering the “past”  $2N - 1$  samples of  $v$  in order to ensure injectivity of the new map  $\Lambda$ . From a least-squares point of view, if the cost function penalizes the deviation from a given output  $y(0)$ , then the controller must be given the chance to affect  $y(0)$ . This can only be guaranteed if we are allowed to choose the  $2N - 1$  inputs preceding that output. This is not

surprising if we recall the  $n$ -step controllability results for discrete-time LTI systems. If the above conditions are satisfied then non-zero inputs sent by the controller to the LTI system will produce non-zero outputs and  $\Lambda$  will be kernel-free. We will not give a formal proof of this here; a more detailed argument can be found in [9].

We expect to include results demonstrating combined tracking and stabilization tasks in the final version of this manuscript.

## 9 Conclusions and Future Work

In this paper we have addressed the stabilization of LTI systems which are operating under limited communication. Our approach is based on the use of periodic communication sequences which direct the flow of control and measurement signals from the controller to the plant across a network or communication bus. The work presented here builds on previous versions of the extensification algorithm in order to handle constraints affecting both control and measurements. There are several issues that present opportunities for further work related to the extensification algorithm, including methods for finding “good” communication sequences and cooling schedules. In addition, it would be desirable to investigate bounds for the spectral radius of the extensive form so that stopping criteria for our simulated annealing algorithm can be constructed.

We have outlined a method for combining basic feed-forward control with the extensive form. Closed-loop tracking is a logical next step that remains to be addressed. Finally, although our model for limited communication assumes timely arrival of each input/output sample from/to the controller, this may not be the case in many realistic situations (i.e. control over the Internet). Current efforts are focused on developing models that can handle variability in the arrivals of control samples as well as “lost” samples that fail to arrive at their destination.

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## References

- [1] V. Blondell and J. Tsitsiklis. NP hardness of some linear control design problems. *SIAM Journal on Control and Optimization*, 35(6):2118–2127, November 1997.
- [2] V. S. Borkar and S. K. Mitter. LQG control with communication constraints. Technical Report LIDS-P-2326, MIT, 1995.
- [3] M. S. Branicky, V. S. Borkar, , and S. K. Mitter. A unified framework for hybrid control: model and optimal control theory. *IEEE Transactions on Automatic Control*, 43(1):31–45, January 1998.
- [4] R. W. Brockett. On the computer control of movement. In *Proceedings of the 1988 IEEE Conference on Robotics and Automation*, pages 534–540, April 1988.
- [5] R. W. Brockett. Hybrid models for motion control systems. In H. Trentelman and J.C. Willems, editors, *Perspectives in Control*, pages 29–54. Birkhäuser, Boston, 1993.
- [6] R. W. Brockett. Stabilization of motor networks. In *Proc. of the 34th IEEE Conference on Decision and Control*, pages 1484–8, December 1995.
- [7] Jon H. Davis. Stability conditions derived from spectral theory: discrete systems with periodic feedback. *SIAM Journal of Control*, 10(1):1–13, February 1972.
- [8] D. Hristu. Generalized inverses for finite-horizon tracking. To appear in *Proc. of the 38 IEEE Conference on Decision and Control*, December 1999.
- [9] D. Hristu. *Optimal Control with Limited Communication*. PhD thesis, Harvard University, Div. of Engineering and Applied Sciences, 1999.
- [10] D. Hristu and K. Morgansen. Limited communication control. *Systems and Control Letters*, 37(4):193–205, July 1999.
- [11] P. Voulgaris. Control of asynchronous sampled data systems. *IEEE Transactions on Automatic Control*, 39(7):1451–1455, July 1994.
- [12] W. S. Wong and R. W. Brockett. Systems with finite bandwidth constraints - part II: Stabilization with limited information feedback. *IEEE Transactions on Automatic Control*, 42(5):1049–1052, May 1999.